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$(C_2 \oplus C_2 \oplus C_2 \oplus C_{2n})!$

by

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1. We prove that any sequence of length 2n + 3 consisting of elements from the Abelian group $G_n = C_2 \oplus C_2 \oplus C_2 \oplus C_{2n}$ (n odd) contains a non empty subsequence with sum zero.
   This result is known for even n (by a result of P. VAN EMDE BOAS). Hence $C_2 \oplus C_2 \oplus C_2 \oplus C_{2n}!$ holds for any n.

   For the general problem, notations and connected results see [1]. The result was already known for n = 3 [2]; we will prove this special case again. As will be seen, it is not a straightforward corollary of the proof for the general case with $n \geq 5$. The statement is trivial for n = 1; we therefore assume $n \geq 3$.

2. The elements of G_n are denoted by volumn vectors $\begin{bmatrix} a \\ x \end{bmatrix}$, with $a \in C_2 \oplus C_2 \oplus C_2 \oplus C_2$ and $x \in C_n$.

   The elements $a \in C_2 \oplus C_2 \oplus C_2 \oplus C_2$ are column vectors thenselves; we use the following fixed designations (designation indicated above the column vector):

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_0
1	0	0	0	1	1	1	0	0	0	1	1	1	0	1	0
0	1	0	0	1	0	0	1	1	0	1	1	0	1	1	0
0	0	1	0	0	1	0	1	0	1	1	0	1	1	1	0
0	0	0	1	0	0	1	0	1	1	0	1	1	1	1	0

3. We use continually the fact that $C_2 \oplus C_2 \oplus C_2 \oplus C_2$ is a vector space over the two-element field $\mathbb{F}_2$; see [1], [2]. Doing this one sees easily:

   (i) Any sequence of length 7 containing at most 3 linear independent elements contains 2 disjoint zero-sequences.

   (ii) A sequence containing no zero-subsequences of length < 4 has length ≤ 8. It is - after a suitable choice of basis elements - contained in the sequence $(a_1,a_2,a_3,a_4,a_{11},a_{12},a_{13},a_{14})$.

(iii) By (ii) any sequence of length ≥ 9 contains a zero-sequence of length ≤ 3. As $9 - 3 = 6 > \lambda(C_2 \oplus C_2 \oplus C_2 \oplus C_2)$ we conclude that any sequence of length ≥ 9 contains at least two disjoint zero-subsequences.

4. Any sequence in $C_2 \oplus C_2 \oplus C_2 \oplus C_2$ of length ≥ 11 contains at least 3 disjoint zero-subsequences.

This follows by 3 (iii) if the length is > 11, or if the sequence contains a_0 or a repetition. Suppose therefore that the sequence consists of 11 distinct elements $\neq 0$. We determine all these sequences (modulo permutations and basis-transforms) by determining all 4-tuple's of distinct non-zero elements and taking their complements afterwards.

We have the following three possibilities:

(i) four linearly independent elements; say a_{11}, a_{12}, a_{13}, a_{14};

(ii) three linearly independent elements and their sum; a_{12}, a_{13}, a_{14}, a_4;

(ii) three linearly independent elements and the sum of two of them; say a_{12}, a_{13}, a_{14}, a_5.

This leads to the following sequences of length 11 consisting of distinct non-zero elements:

(i)' $(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{15})$.
Disjoint zero-sequences: $a_1 + a_2 + a_5$, $a_3 + a_4 + a_{10}$, $a_6 + a_7 + a_8 + a_9$.

(ii)' $(a_1,a_2,a_3,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{15})$.
Disjoint zero-sequences: $a_1 + a_2 + a_5$, $a_6 + a_7 + a_{10}$, $a_3 + a_8 + a_9 + a_{11} + a_{15}$.

(iii)' $(a_1,a_2,a_3,a_4,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{15})$.
Disjoint zero-sequences: $a_1 + a_3 + a_6$, $a_2 + a_7 + a_9$, $a_7 + a_8 + a_{15}$.

5. Any sequence in $C_2 \oplus C_2 \oplus C_2 \oplus C_2$ with length ≥ 13 contains at least 4 disjoint zero-subsequences.

This follows by 4. if the sequence contains a_0 or a repetition; if the sequence contains 13 distinct non-zero elements, say a_1 up to a_{13} we have the zero-sequences:

$a_2 + a_4 + a_9$, $a_3 + a_5 + a_{11}$, $a_6 + a_7 + a_{10}$, $a_8 + a_{12} + a_{13}$.

6. Any sequence in $C_2 \oplus C_2 \oplus C_2 \oplus C_2$ with length ≥ 15 contains at least 5 disjoint zero-subsequences.

This follows by 5 if the sequence contains a_0 or a repetition; if the sequence contains 15 distinct non-zero elements, i.e. a_1 up to a_{15} we have the zero-sequences:

$a_1 + a_2 + a_5$, $a_3 + a_4 + a_{10}$, $a_6 + a_9 + a_{15}$, $a_7 + a_{11} + a_{14}$, $a_8 + a_{12} + a_{13}$.

7. We need to know all the possible sequences of nine distinct non-zero elements. Hence we first consider (modulo basis transforms) all possible combinations of six distinct non-zero elements:

(i) There are no 4 linearly independent elements; the only possible type now is $(a_1,a_2,a_3,a_5,a_6,a_8)$.

(ii) There are 4 independent elements together with their sum; type $(a_1\ a_2\ a_3\ a_4\ a_{11}\ a_{15})$
(which is equivalent with $(a_1\ a_2\ a_3\ a_4\ a_5\ a_{15})$).

(iii) 4 independent elements, no sum of 4 or 2 elements; type $(a_1\ a_2\ a_3\ a_4, a_{11}, a_{12})$.

(iv) 4 independent elements, no sum of 4 elements, sums of three and two elements;
type $(a_1,a_2,a_3,a_4,a_5,a_{11})$
$(a_1,a_2,a_3,a_4,a_7,a_{11})$ is equivalent with (ii).

(v) 4 independent elements, no sum of 3 or 4 elements;
type $(a_1,a_2,a_3,a_4,a_5,a_{10})$
$(a_1,a_2,a_3,a_4,a_5,a_6)$ is equivalent with (iv).

We may take the following equivalent sequences:

(i)' $(a_{14},a_{13},a_{12},a_5,a_6,a_8)$;

(ii)' $(a_{14},a_{13},a_{12},a_{11},a_{15},a_{10})$;

(iii)' $(a_{15},a_{14},a_{13},a_{12},a_{10},a_9)$;

(iv)' $(a_{15},a_{14},a_{13},a_{12},a_{10},a_1)$;

(v)' $(a_{14},a_{13},a_{12},a_{11},a_5,a_{10})$.

By passing to the complements we get a complete set fo representatives for the collections of 9 distinct non-zero elements in $C_2 \oplus C_2 \oplus C_2 \oplus C_2$:

(i)" $(a_1,a_2,a_3,a_4,a_7,a_9,a_{10},a_{11},a_{15})$;

(ii)" $(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9)$;

(iii)" $(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_{11})$;

(iv)" $(a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{11})$;

(v)" $(a_1,a_2,a_3,a_4,a_6,a_7,a_8,a_9,a_{15})$.

(v)" contains three disjoint zero-subsequences:
$a_1 + a_3 + a_6$, $a_2 + a_4 + a_9$, $a_7 + a_8 + a_{15}$.

8. We need a survey on the possible subcollections of 7 distinct non-zero elements from $C_2 \oplus C_2 \oplus C_2 \oplus C_2$. Starting with the collections (i) - (v) from 7, we find the following possible extensions:

From (i) (1) $(a_1,a_2,a_3,a_5,a_6,a_8,a_{11})$
(all other extensions contain 4 independent elements).

From (ii) (2) $(a_1,a_2,a_3,a_4,a_{15},a_{11},a_{12})$;
(3) $(a_1,a_2,a_3,a_4,a_{15},a_{11},a_5)$;
(4) $(a_1,a_2,a_3,a_4,a_{15},a_{11},a_{10})$.

From (iii) (5) $(a_1,a_2,a_3,a_4,a_{11},a_{12},a_{13})$;
(6) $(a_1,a_2,a_3,a_4,a_{11},a_{12},a_5)$;
(7) $(a_1,a_2,a_3,a_4,a_{11},a_{12},a_{10})$;
(8) $(a_1,a_2,a_3,a_4,a_{11},a_{12},a_8)$.

From (iv) (9) $(a_1,a_2,a_3,a_4,a_5,a_{11},a_6)$;
(10) $(a_1,a_2,a_3,a_4,a_5,a_{11},a_7)$;
(11) $(a_1,a_2,a_3,a_4,a_5,a_{11},a_{10})$.

From (v) (12) $(a_1,a_2,a_3,a_4,a_5,a_{10},a_6)$.

Some of these combinations are equivalent:
By taking the basis (a_1,a_2,a_3,a_{15}) (4) is transformed into (2).
From (5) up to (12) we may reject all combinations containing the sum of 4 independent elements as these are equivalent to (2) or (3). For example in (12) we have $a_4 = a_2 + a_5 + a_6 + a_{10}$; hence (12) is equivalent to (2) or (3). (In fact (12) $\sim$ (3) by taking the basis (a_2,a_5,a_{10},a_6)).
The same happens in (7), (8), (10) and (11).
$(a_1+a_2+a_3+a_{12}=a_{10},\ a_1+a_3+a_4+a_{12}=a_8,\ a_2+a_3+a_4+a_7=a_{11},\ a_1+a_2+a_4+a_{10}=a_{11})$.

The remaining sequences are

(i) $(a_1,a_2,a_3,a_4,a_{11},a_{12},a_{15}) = (2)$;

(ii) $(a_1,a_2,a_3,a_4,a_5,a_6,a_{11}) = (9)$;

(iii) $(a_1,a_2,a_3,a_4,a_{11},a_{12},a_{13}) = (5)$;

(iv) $(a_1,a_2,a_3,a_5,a_6,a_8,a_{11}) = (1)$;

(v) $(a_1,a_2,a_3,a_4,a_5,a_{11},a_{15}) = (3)$;

(vi) $(a_1,a_2,a_3,a_4,a_5,a_{11},a_{12}) = (6)$.

(iv), (v) and (vi) each contain two disjoint zero-subsequences:

In (iv) $a_1 + a_2 + a_5$, $a_3 + a_6 + a_8 + a_{11}$;

In (v) $a_1 + a_2 + a_5$, $a_4 + a_{11} + a_{15}$;

In (vi) $a_1 + a_2 + a_5$, $a_3 + a_4 + a_{11} + a_{12}$.

These six sentences form a complete set of representatives of all possible sequences of length 7 consisting of distinct non-zero elements.

9. In C_n any sequence of length n - 1, without zero-subsequences consists of a fixed generator g of C_n taken n - 1 times. For a sequence of length n - 2 without zero-subsequences there are two possibilities: either the sequence consists of a fixed generator g taken n - 2 times, or the sequence contains a generator g n - 3 times and the element 2g exactly once.

10. A <u>short zero-sequence</u> in $C_2 \oplus C_2 \oplus C_2 \oplus C_2$ is a zero-sequence of length 1 or 2. Any $C_2 \oplus C_2 \oplus C_2 \oplus C_2$ - sequence of length ≥ 16 contains a short zero-subsequence; hence any sequence of length 2n + 3 contains at least n - 6 short zero-subsequences while the remaining elements (at least 15) contain another set of 5 disjoint zero-subsequences. Together this gives at least n - 1 zero-sequences.

Let S be a G_n-sequence of length 2n + 3 and let π be the projection from G_n onto $C_2 \oplus C_2 \oplus C_2 \oplus C_2$. Let $A = \pi(S)$.
If A contains n disjoint zero-sequences, then S itself contains a zero-sequence, as $n > \lambda\, C_n$.

<u>From now on we assume that A contains only n - 1 disjoint zero-sequences</u>, say $\pi S_1, \ldots, \pi S_{n-1}$. The $|S_1|, \ldots, |S_{n-1}|$ are n - 1 elements in a subgroup $H \subset G_n$ which is isomorphic to C_n. These elements generate a zero-sequence except for the case that $|S_1| = |S_2| = \ldots = |S_{n-1}| = g$ for some generator g of H.
(As $H = (\begin{bmatrix} a \\ x \end{bmatrix} \in G_n \mid a = a_0)$ we identify H and C_n).
As n is an odd number, we may take g = 4.

Within A there are the following possibilities:

(α) there are $n - 6$ disjoint short zero-sequences + 15 distinct non-zero elements;

(β) there are $n - 5$ disjoint short zero-sequences + at least 13 distinct non-zero elements;

(γ) there are $n - 4$ disjoint short zero-sequences + at least 11 distinct non-zero elements;

(δ) there are $n - 3$ disjoint short zero-sequences + at least 9 distinct non-zero elements;

(ε) there are $n - 2$ disjoint short zero-sequences + at least 7 distinct non-zero elements.

The case with $n - 1$ disjoint short zero-sequences + at least 5 distinct non-zero elements is not interesting as these 5 elements contain another zero-sequence; hence A then contains n zero-subsequences.

N.B. For small n some of these cases may be absent!

Next we treat each of these five cases seperately.

11. In the sequel we may assume that all x_i occuring as C_n-coördinate of an element from S, are different from zero.
For suppose that S contains an element $\begin{bmatrix} b_i \\ 0 \end{bmatrix}$.

We may put $b_i = a_1$. As $2n + 2 > \lambda\ (C_2 \oplus C_2 \oplus C_{2n})$ we are sure that the remaining $2n + 2$ elements contain a subsequence T with sum $\begin{bmatrix} a \\ 0 \end{bmatrix}$, where $a + a_0$ or a_1. Indeed, let ρ be the projection
$\rho\colon C_2 \oplus C_2 \oplus C_2 \oplus C_{2n} \to C_2 \oplus C_2 \oplus C_{2n}$ defined by

$$\rho\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ x \end{bmatrix}\right) = \begin{bmatrix} a_2 \\ a_3 \\ a_4 \\ x \end{bmatrix}.$$

The remaining 2n + 2 elements are mapped by ρ on a sequence which contains a zero-sequence ρT. Now T has a sum which is contained in kernel of ρ, that is, in $\{ \begin{bmatrix} a_0 \\ 0 \end{bmatrix}, \begin{bmatrix} a_1 \\ 0 \end{bmatrix} \}$.

But then either T or $T \cup \{ \begin{bmatrix} a_1 \\ 0 \end{bmatrix} \}$ is a zero-sequence.

12. Case α.

Case α has meaning only for $n \geq 7$.
Any zero-sequence of length 3 in A is member of a set of 5 disjoint zero-subsequences in A. Hence any sequence S' of length 3 with $|\pi S| = 0$ must satisfy $|S| = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$. Let S consist of n - 6 short zero-sequences and the elements

$$\begin{bmatrix} a_1 \\ x_1 \end{bmatrix} \begin{bmatrix} a_2 \\ x_2 \end{bmatrix} \begin{bmatrix} a_3 \\ x_3 \end{bmatrix} \begin{bmatrix} a_4 \\ x_4 \end{bmatrix} \begin{bmatrix} a_5 \\ x_5 \end{bmatrix} \begin{bmatrix} a_6 \\ x_6 \end{bmatrix} \begin{bmatrix} a_7 \\ x_7 \end{bmatrix} \begin{bmatrix} a_8 \\ x_8 \end{bmatrix} \begin{bmatrix} a_9 \\ x_9 \end{bmatrix} \begin{bmatrix} a_{10} \\ x_{10} \end{bmatrix} \begin{bmatrix} a_{11} \\ x_{11} \end{bmatrix} \begin{bmatrix} a_{12} \\ x_{12} \end{bmatrix} \begin{bmatrix} a_{13} \\ x_{13} \end{bmatrix} \begin{bmatrix} a_{14} \\ x_{14} \end{bmatrix} \begin{bmatrix} a_{15} \\ x_{15} \end{bmatrix}.$$

This gives us the following relations:

$$\boxed{\text{I}} \qquad \begin{aligned} & x_2 + x_3 + x_8 = x_2 + x_4 + x_9 = x_2 + x_6 + x_{11} = \\ & x_2 + x_7 + x_{12} = x_3 + x_4 + x_{10} = x_4 + x_5 + x_{12} = \\ & x_4 + x_6 + x_{13} = x_5 + x_6 + x_8 = x_5 + x_7 + x_9 = \\ & x_8 + x_9 + x_{10} = 4. \end{aligned}$$

From these we derive:

$$(x_2+x_3+x_8) + (x_2+x_4+x_9) + (x_3+x_4+x_{10}) = 8 + (x_8+x_9+x_{10})$$

hence $x_2 + x_3 + x_4 = 4$, i.e. $x_8 = x_4$, $x_9 = x_3$, $x_{10} = x_2$

and $x_4 + x_6 + x_{13} = x_4 + x_6 + x_5 = 4$, i.e. $x_5 = x_{13}$.

Furthermore $(x_2+x_7+x_{12}) + (x_4+x_6+x_5) = (x_4+x_5+x_{12}) + 4$

hence $x_2 + x_6 + x_7 = 4$; i.e. $x_6 = x_{12}$, $x_7 = x_{11}$.

Finally $(x_2+x_3+x_4) + (x_2+x_6+x_7) + (x_4+x_5+x_6) = (x_5+x_7+x_3) + 8$;

Hence $2(x_2+x_4+x_6) = 8$, i.e. $x_2 + x_4 + x_6 = 4$.

Therefore $x_6 = x_3$, $x_5 = x_2$, $x_7 = x_4$.

Conclusion:

$\boxed{\text{II}}$
$$\begin{aligned} x_2 &= x_5 = x_{10} = x_{13}; \\ x_3 &= x_6 = x_9 = x_{12}; \\ x_4 &= x_7 = x_8 = x_{11}; \\ x_2 &+ x_3 + x_4 = 4 \ . \end{aligned}$$

If $x_2 = 0$ or $x_3 = 0$ or $x_4 = 0$, S contains a zero-sequence, as $a_2 + a_5 + a_{10} + a_{13} = a_3 + a_6 + a_9 + a_{12} = a_4 + a_7 + a_8 + a_{11} = 0$. Hence we may assume $x_2 \not\equiv 0$, $x_3 \not\equiv 0$, $x_4 \not\equiv 0$.

The zero-sequence a_2, a_3, a_5, a_6 is member of a family of $n - 2$ disjoint zero-subsequences (containing the $n - 6$ short zero-subsequences and 3 in the remaining 11 elements a_1, a_4, a_7 up to a_{15}). Hence $x_2 + x_3 + x_5 + x_6 \in \{4,8\}$, i.e. $x_2 + x_3 \in \{2,4\}$. Simmilarly $x_6 + x_7 + x_8 + x_9 \in \{4,8\}$, i.e. $x_3 + x_4 \in \{2,4\}$ and $x_2 + x_4 + x_8 + x_{10} \in \{4,8\}$, i.e. $x_2 + x_4 \in \{2,4\}$.

Now $x_2 + x_3 = 4 \Rightarrow x_4 = 0$ quod non; hence $x_2 + x_3 = 2$. Simmilarly $x_2 + x_4 = x_3 + x_4 = 2$. Hence $x_2 = x_3 = x_4 = 1$. This is a contradiction as $3 \not\equiv 4 \pmod{n}$.

13. Case β.

Case β has meaning only for $n \geq 5$.
Let A consist of $n - 5$ disjoint short zero-sequences and the 13 elements a_1, up to a_{13}.

as $\sum_{i=2}^{13} a_i = 0$, any zero-subsequence of length 3 in $(a_2,\dots,a_{13})$ is member of a family of 4 disjoint zero-subsequences (by 3(iii) and $6 > \lambda(C_2 \oplus C_2 \oplus C_2 \oplus C_2) + 1$).

This leads us once again to the equations $\boxed{\mathrm{I}}$, which have been proved to be contradicting in case α. These equations can be derived also in the case $n = 5$, although then there are no short zero-sequences in A.

14. Case γ.

Case γ has meaning only for $n \geq 5$; hence there exists a short zero-sequence. By (4) there are three subcases, depending on the type of the sequence of the remaining 11 elements:

γ(i). Remaining elements:

$$\begin{bmatrix} a_1 \\ x_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ x_2 \end{bmatrix}, \begin{bmatrix} a_3 \\ x_3 \end{bmatrix}, \begin{bmatrix} a_4 \\ x_4 \end{bmatrix}, \begin{bmatrix} a_5 \\ x_5 \end{bmatrix}, \begin{bmatrix} a_6 \\ x_6 \end{bmatrix}, \begin{bmatrix} a_7 \\ x_7 \end{bmatrix}, \begin{bmatrix} a_8 \\ x_8 \end{bmatrix}, \begin{bmatrix} a_9 \\ x_9 \end{bmatrix}, \begin{bmatrix} a_{10} \\ x_{10} \end{bmatrix}, \begin{bmatrix} a_{15} \\ x_{15} \end{bmatrix}.$$

Now $\sum_{i=1}^{10} a_i$ is a zero-sequence. Hence any zero-subsequence of $(a_1,\dots,a_{10})$ of length 3 or 4 is member of a family of three disjoint zero-subsequences.

This gives the following equations:

$\boxed{\mathrm{III}}$

$$x_1 + x_2 + x_5 = x_7 + x_9 + x_5 = x_1 + x_2 + x_7 + x_9 = 4$$
$$\text{i.e. } x_1 + x_2 = x_7 + x_9 = x_5 = 2$$

$$x_1 + x_2 + x_5 = x_1 + x_3 + x_6 = x_2 + x_5 + x_3 + x_6 = 4$$
$$\text{i.e. } x_2 + x_5 = x_3 + x_6 = x_1 = 2$$

$$x_1 + x_2 + x_5 = x_2 + x_3 + x_8 = x_1 + x_5 + x_3 + x_8 = 4$$
$$\text{i.e. } x_1 + x_5 = x_3 + x_8 = x_2 = 2$$

The result $x_1 + x_2 = x_1 = x_2 = 2$ is impossible.

γ(ii). Remaining elements:

$$\begin{bmatrix} a_1 \\ x_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ x_2 \end{bmatrix}, \begin{bmatrix} a_3 \\ x_3 \end{bmatrix}, \begin{bmatrix} a_5 \\ x_5 \end{bmatrix}, \begin{bmatrix} a_6 \\ x_6 \end{bmatrix}, \begin{bmatrix} a_7 \\ x_7 \end{bmatrix}, \begin{bmatrix} a_8 \\ x_8 \end{bmatrix}, \begin{bmatrix} a_9 \\ x_9 \end{bmatrix}, \begin{bmatrix} a_{10} \\ x_{10} \end{bmatrix}, \begin{bmatrix} a_{11} \\ x_{11} \end{bmatrix}, \begin{bmatrix} a_{15} \\ x_{15} \end{bmatrix}.$$

Now the whole sequence is a zero-sequence and the equations [III] can be derived again.

γ(iii). Remaining elements:

$$\begin{bmatrix} a_1 \\ x_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ x_2 \end{bmatrix}, \begin{bmatrix} a_3 \\ x_3 \end{bmatrix}, \begin{bmatrix} a_4 \\ x_4 \end{bmatrix}, \begin{bmatrix} a_6 \\ x_6 \end{bmatrix}, \begin{bmatrix} a_7 \\ x_7 \end{bmatrix}, \begin{bmatrix} a_8 \\ x_8 \end{bmatrix}, \begin{bmatrix} a_9 \\ x_9 \end{bmatrix}, \begin{bmatrix} a_{10} \\ x_{10} \end{bmatrix}, \begin{bmatrix} a_{11} \\ x_{11} \end{bmatrix}, \begin{bmatrix} a_{15} \\ x_{15} \end{bmatrix}.$$

This time we don't have a collection of three disjoint zero-sequences containing a zero-sequence of length 4. By 3(iii) any zero-sequence of length 9 contains three zero-subsequences of length 3. By searching all pairs of elements with sum $a_{12} = a_1 + a_2 + a_3 + a_4 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{15}$ we can find all these zero-sequences of length 9.
We have $a_{12} = a_1 + a_9 = a_2 + a_7 = a_3 + a_{15} = a_{10} + a_{11}$. From this we conclude that any zero-sequence of length 3 is member of a family of three disjoint zero-subsequences of length 3, for a sequence of three element cannot contain a member in each of the 4 pairs $\{a_1,a_9\},\{a_2,a_7\},\{a_3,a_{15}\}$ and $\{a_{10},a_{11}\}$.

Hence we have the following equations:

[IV]
$$\begin{aligned} & x_1 + x_3 + x_6 = x_1 + x_4 + x_7 = x_1 + x_8 + x_{11} = \\ & x_2 + x_3 + x_8 = x_2 + x_4 + x_9 = x_2 + x_6 + x_{11} = \\ & x_3 + x_4 + x_{10} = x_4 + x_{11} + x_{15} = x_6 + x_7 + x_{10} = \\ & x_6 + x_9 + x_{15} = x_8 + x_9 + x_{10} = 4. \end{aligned}$$

From $x_2 + x_3 + x_8 = x_2 + x_4 + x_9 = x_3 + x_4 + x_{10} = x_8 + x_9 + x_{10} = 4$

we derive as in 12: $x_2 = x_{10}$, $x_3 = x_9$, $x_4 = x_8$, $x_2 + x_3 + x_4 = 4$.

Simmilarly $x_1 + x_3 + x_6 = x_1 + x_4 + x_7 = x_3 + x_4 + x_{10} = x_6 + x_7 + x_{10} = 4$

gives $x_6 = x_4$, $x_7 = x_3$, $x_{10} = x_1$.

The remaining equations become:

$$x_1 + x_4 + x_{11} = 4 \qquad \text{i.e.} \quad x_{11} = x_3;$$
$$x_4 + x_3 + x_{15} = 4 \qquad \text{i.e.} \quad x_{15} = x_1.$$

Hence $x_1 = x_2 = x_{10} = x_{15}$, $x_3 = x_7 = x_9 = x_{11}$, $x_4 = x_6 = x_8$.

By considering zero-subsequences of length 4 we have

$$x_1 + x_3 + x_7 + x_{10} \in \{4,8\} \quad \text{(disjoint with } (a_2,a_4,a_9))$$
$$x_1 + x_2 + x_6 + x_8 \in \{4,8\} \quad \text{(disjoint with } (a_3,a_4,a_{10}))$$
$$x_3 + x_4 + x_8 + x_9 \in \{4,8\} \quad \text{(disjoint with } (a_6,a_7,a_{10}))$$

Hence $x_2 + x_3$, $x_2 + x_4$ and $x_3 + x_4 \in \{2,4\}$

Now $a_1 + a_2 + a_{10} + a_{15} = a_3 + a_7 + a_9 + a_{11} = a_0$

Hence $4 \cdot x_1 \not\equiv 0$ and $4 \cdot x_3 \not\equiv 0$.

Therefore $2 = x_2 + x_4 \not\equiv 4$ and $2 = x_3 + x_4 \not\equiv 4$ and it follows that $x_2 = x_3$.

Now either $x_2 = x_3 = 1$ and $x_4 = 2$ which leads to the contradicting $2 = x_2 + x_4 = 3$, or $x_2 = x_3 = 2$ and $x_4 = 0$.

As (a_1,a_2,a_{10},a_{15}) and (a_3,a_7,a_9,a_{11}) are disjoint we have

$x_1 + x_2 + x_{10} + x_{15} = 4\, x_2 \in \{4,8\}$ and
$x_3 + x_7 + x_9 + x_{11} = 4\, x_3 \in \{4,8\}$, but not simultaneously
$4\, x_2 = 8 = 4\, x_3$ (see 9)!

Thus the equations lead to a contradiction.

15. Case δ.

Case δ has meaning for $n \geq 3$. However for $n = 3$ there is no short zero-sequence in A. Therefore we may conclude equality for two x-values of subsequences of S with A-value zero only if they are disjoint.

From 7 we conclude that we have 5 different subcases depending of the type of the 9 remaining elements. In case 7 (v)" the remaining part of A contains three disjoint zero-subsequences; hence A contains n disjoint zero-subsequences, and we conclude that S contains a zero-sequence.

Next we treat the four remaining subcases.

δ(i)" Remaining elements:

$$\begin{bmatrix} a_1 \\ x_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ x_2 \end{bmatrix}, \begin{bmatrix} a_3 \\ x_3 \end{bmatrix}, \begin{bmatrix} a_4 \\ x_4 \end{bmatrix}, \begin{bmatrix} a_7 \\ x_7 \end{bmatrix}, \begin{bmatrix} a_9 \\ x_9 \end{bmatrix}, \begin{bmatrix} a_{10} \\ x_{10} \end{bmatrix}, \begin{bmatrix} a_{11} \\ x_{11} \end{bmatrix}, \begin{bmatrix} a_{15} \\ x_{15} \end{bmatrix}.$$

We have the following equations:

$$x_3+x_{10}+x_{11}+x_{15} = x_1+x_4+x_7 = x_2+x_3+x_9+x_{10} = x_4+x_{11}+x_{15} = 4;$$
$$x_1+x_7+x_{11}+x_{15} = x_2+x_4+x_9 = x_1+x_3+x_7+x_{10} = x_4+x_{11}+x_{15} = 4;$$
$$x_2+x_9+x_{11}+x_{15} = x_3+x_4+x_{10} = x_1+x_2+x_7+x_9 = x_4+x_{11}+x_{15} = 4.$$

Hence $x_4 = x_1 + x_7 = x_2 + x_9 = x_3 + x_{10} = x_{11} + x_{15}$.

Furthermore we have

$$x_1 + x_2 + x_{10} + x_{15} = x_3 + x_7 + x_9 + x_{11} = r_1,$$
$$x_1 + x_3 + x_9 + x_{15} = x_2 + x_7 + x_{10} + x_{11} = r_2,$$
$$x_2 + x_3 + x_7 + x_{15} = x_1 + x_9 + x_{10} + x_{11} = r_3, \quad \text{and}$$
$$x_1 + x_2 + x_3 + x_{11} = x_7 + x_9 + x_{10} + x_{15} = r_4.$$

Hence $x_1 + x_2 + x_3 + 3\,x_{15} = x_7 + x_9 + x_{10} + 3\,x_{11}$.

Therefore $4\,x_{15} = 4\,x_{11}, \Rightarrow x_{15} = x_{11} = 1,\ x_4 = 2$.

Now we may write $x_1 = a, \quad x_2 = b, \quad x_3 = c,$

$$x_7 = 2 - a,\ x_9 = 2 - b,\ x_{10} = 2 - c.$$

If $n = 3$ we have $a \neq 0 \neq 2 - a$, hence $a = 2 - a = 1$;
and in the same way: $b = 1$, $c = 1$.

For $n > 3$ we have $r_1 = 4 = r_4$, hence $a + b + c = a + b + (2-c)$,
and we conclude again that $c = 1$, and similarly that $a = b = 1$.

In both cases we have $x_3+x_4+x_7+x_9+x_{15} = 6 \notin \{4,8\}$, which shows that S admits a zero-subsequence as $a_3 + a_4 + a_7 + a_9 + a_{15} = a_0$.

δ(ii)" Remaining elements:

$$\begin{bmatrix} a_1 \\ x_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ x_2 \end{bmatrix}, \begin{bmatrix} a_3 \\ x_3 \end{bmatrix}, \begin{bmatrix} a_4 \\ x_4 \end{bmatrix}, \begin{bmatrix} a_5 \\ x_5 \end{bmatrix}, \begin{bmatrix} a_6 \\ x_6 \end{bmatrix}, \begin{bmatrix} a_7 \\ x_7 \end{bmatrix}, \begin{bmatrix} a_8 \\ x_8 \end{bmatrix}, \begin{bmatrix} a_9 \\ x_9 \end{bmatrix}.$$

We have the following equations:

$$\begin{array}{ccccccc}
x_2 + x_4 + x_5 + x_7 & = & x_1 + x_3 + x_6 & = & x_2 + x_4 + x_9 & = & x_1 + x_3 + x_5 + x_8, \\
 & & \parallel & & \parallel & & \\
x_1 + x_2 + x_6 + x_8 & = & x_5 + x_7 + x_9 & & x_5 + x_6 + x_8 & = & x_1 + x_2 + x_7 + x_9, \\
 & & \parallel & & \parallel & & \\
x_1 + x_4 + x_5 + x_9 & = & x_2 + x_3 + x_8 & = & x_1 + x_4 + x_7 & = & x_2 + x_3 + x_5 + x_6.
\end{array}$$

Hence we conclude that
$$\begin{aligned} x_2 &= x_1 + x_5, \\ x_1 &= x_2 + x_5, \\ x_5 &= x_1 + x_2. \end{aligned}$$

Therefore $x_1 + x_2 + x_5 = 0$. Thus S has a zero-subsequence, as $a_1 + a_2 + a_5 = a_0$.

δ(iii)" Remaining elements:

$$\begin{bmatrix} a_1 \\ x_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ x_2 \end{bmatrix}, \begin{bmatrix} a_3 \\ x_3 \end{bmatrix}, \begin{bmatrix} a_4 \\ x_4 \end{bmatrix}, \begin{bmatrix} a_5 \\ x_5 \end{bmatrix}, \begin{bmatrix} a_6 \\ x_6 \end{bmatrix}, \begin{bmatrix} a_7 \\ x_7 \end{bmatrix}, \begin{bmatrix} a_8 \\ x_8 \end{bmatrix}, \begin{bmatrix} a_{11} \\ x_{11} \end{bmatrix}.$$

We have the following equations:

$$x_2 + x_3 + x_5 + x_6 = x_1 + x_4 + x_7 = x_3 + x_5 + x_{11},$$
$$x_1 + x_5 + x_6 + x_{11} = x_2 + x_3 + x_8 = x_1 + x_2 + x_5 ,$$
$$x_1 + x_2 + x_3 + x_{11} = x_5 + x_6 + x_8 = x_1 + x_4 + x_7 =$$
$$x_2 + x_3 + x_5 + x_6 = x_1 + x_8 + x_{11} = x_2 + x_4 + x_5 + x_7 =$$
$$x_1 + x_3 + x_6.$$

Therefore $x_2 + x_6 = x_{11}$, $x_6 + x_{11} = x_2$, $x_2 + x_{11} = x_6$;

Hence $x_2 + x_6 + x_{11} = 0$, which leads to a zero-subsequence as $a_2 + a_6 + a_{11} = a_0$.

δ(iv)" Remaining elements:

$$\begin{bmatrix} a_2 \\ x_2 \end{bmatrix}, \begin{bmatrix} a_3 \\ x_3 \end{bmatrix}, \begin{bmatrix} a_4 \\ x_4 \end{bmatrix}, \begin{bmatrix} a_5 \\ x_5 \end{bmatrix}, \begin{bmatrix} a_6 \\ x_6 \end{bmatrix}, \begin{bmatrix} a_7 \\ x_7 \end{bmatrix}, \begin{bmatrix} a_8 \\ x_8 \end{bmatrix}, \begin{bmatrix} a_9 \\ x_9 \end{bmatrix}, \begin{bmatrix} a_{11} \\ x_{11} \end{bmatrix}.$$

We have the following equations:

$$x_4 + x_6 + x_9 + x_{11} = x_2 + x_3 + x_8 = x_5 + x_7 + x_9 = x_2 + x_6 + x_{11}$$
$$\|$$
$$x_2 + x_4 + x_5 + x_7 = x_3 + x_6 + x_8 + x_{11}$$

Hence $x_2 + x_4 = x_9$ and $x_2 = x_4 + x_9$, i.e.

$x_2 = x_4 + x_4 + x_2$ and $x_4 = 0$.

By 11 the sequence contains a zero-subsequence.

16. Case ε.

Case ε has meaning for $n \geq 3$. Even for $n = 3$ there exists one short zero-sequence in A. Hence we may conclude equality for the x-values of two subsequences having A-value zero.
By 8 there are six possibilities for the remaining seven elements. We need only to consider the first three of them as in the other cases A contains n disjoint zero-subsequences.

The first two of them can be treated as before; in case ε(iii) there are unexpected difficulties.

ε(i) Remaining elements:

$$\begin{bmatrix} a_1 \\ x_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ x_2 \end{bmatrix}, \begin{bmatrix} a_3 \\ x_3 \end{bmatrix}, \begin{bmatrix} a_4 \\ x_4 \end{bmatrix}, \begin{bmatrix} a_{11} \\ x_{11} \end{bmatrix}, \begin{bmatrix} a_{12} \\ x_{12} \end{bmatrix}, \begin{bmatrix} a_{15} \\ x_{15} \end{bmatrix}.$$

We have the following equations:

$$\begin{aligned}
x_3+x_{12}+x_4+x_{11} &= x_3+x_{12}+x_{15} = x_4+x_{11}+x_{15} = 4; \text{ hence } x_{15} = 2;\\
x_3+x_{12}+x_4+x_{11} &= x_1+x_2+x_4+x_{12} = x_1+x_2+x_3+x_{11} = 4; \text{ hence } x_1+x_2 = 2;\\
x_3+x_{12}+x_4+x_{11} &= x_1+x_2+x_{15}+x_{11}+x_{12} = x_1+x_2+x_{15}+x_3+x_4 = 4; \text{ hence } x_1+x_2+x_{15} = 2.
\end{aligned}$$

This is contradictary.

ε(ii) Remaining elements:

$$\begin{bmatrix} a_1 \\ x_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ x_2 \end{bmatrix}, \begin{bmatrix} a_3 \\ x_3 \end{bmatrix}, \begin{bmatrix} a_4 \\ x_4 \end{bmatrix}, \begin{bmatrix} a_5 \\ x_5 \end{bmatrix}, \begin{bmatrix} a_6 \\ x_6 \end{bmatrix}, \begin{bmatrix} a_{11} \\ x_{11} \end{bmatrix}.$$

We have the following equations:

$$\begin{aligned}
x_3+x_6+x_1 &= x_3+x_6+x_2+x_5 = x_1+x_2+x_5 = 4; \text{ hence } x_3+x_6 = 2;\\
x_3+x_6+x_1 &= x_3+x_1+x_2+x_{11} = x_6+x_2+x_{11} = 4; \text{ hence } x_6 = 2;\\
x_3+x_6+x_1 &= x_3+x_5+x_{11} = x_6+x_1+x_5+x_{11} = 4; \text{ hence } x_3 = 2.
\end{aligned}$$

This is contradictary.

ε(iii) Remaining elements:

$$\begin{bmatrix} a_1 \\ x_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ x_2 \end{bmatrix}, \begin{bmatrix} a_3 \\ x_3 \end{bmatrix}, \begin{bmatrix} a_4 \\ x_4 \end{bmatrix}, \begin{bmatrix} a_{11} \\ x_{11} \end{bmatrix}, \begin{bmatrix} a_{12} \\ x_{12} \end{bmatrix}, \begin{bmatrix} a_{13} \\ x_{13} \end{bmatrix}.$$

This case turns out to be the most nasty of them all, calling for elaborate considerations; let's call it the CASUS PERDIFFICILIS.

In particular, we are again forced to consider the cases n = 3 and $n \geq 5$ seperately.

In either case we have the following equations:

$$x_1 + x_2 + x_3 + x_{11} = x_1 + x_2 + x_4 + x_{12} = x_1 + x_3 + x_4 + x_{13} =$$
$$x_3 + x_4 + x_{11} + x_{12} = x_2 + x_3 + x_{12} + x_{13} = x_2 + x_4 + x_{11} + x_{13} =$$
$$x_1 + x_{11} + x_{12} + x_{13} = 4.$$

Hence $x_1 = x_2 = x_3 = x_4 = x_{11} = x_{12} = x_{13} = 1$.

(For any i add the four terms containing x_i and subtract the three terms not containing x_i; result: $4\ x_i = 4$).

17. CASUS PERDIFFICILIS, n = 3.

Now we have exactly one short zero-subsequence in A.

(1) This short zero-sequence has length 1. Then the remaining elements contain an eigth element $\begin{bmatrix} a \\ x \end{bmatrix}$.

(1.1) If $a = a_0$ or $a \in \{a_1, a_2, a_3, a_4, a_{11}, a_{12}, a_{13}\}$ we have two short zero-sequences in A and therefore A contains three disjoint zero-subsequences and S contains a zero-sequence also.

(1.2) $a = a_{14}$. Now as before we derive x = 1 (by interchanging a_{13} and a_{14}).

Hence S is equivalent to:

$$\left(\begin{bmatrix} a_0 \\ 4 \end{bmatrix}, \begin{bmatrix} a_1 \\ 1 \end{bmatrix}, \begin{bmatrix} a_2 \\ 1 \end{bmatrix}, \begin{bmatrix} a_3 \\ 1 \end{bmatrix}, \begin{bmatrix} a_4 \\ 1 \end{bmatrix}, \begin{bmatrix} a_{11} \\ 1 \end{bmatrix}, \begin{bmatrix} a_{12} \\ 1 \end{bmatrix}, \begin{bmatrix} a_{13} \\ 1 \end{bmatrix}, \begin{bmatrix} a_{14} \\ 1 \end{bmatrix}\right)$$

but this is a (primitive!) zero-sequence.

(1.3) $a \notin \{a_0, a_1, a_2, a_3, a_4, a_{11}, a_{12}, a_{13}, a_{14}\}$.

By 3 (ii) the remaining 8 elements contain a zero-sequence of length 3. As there is another zero-sequence among the 5 other elements, A contains three disjoint zero-subsequences and we are done.

(2) The short zero-sequence has length 2. Now S is of the type

$$\begin{bmatrix} a \\ x' \end{bmatrix} \begin{bmatrix} a \\ x'' \end{bmatrix} \begin{bmatrix} a_1 \\ x_1 \end{bmatrix} \begin{bmatrix} a_2 \\ x_2 \end{bmatrix} \begin{bmatrix} a_3 \\ x_3 \end{bmatrix} \begin{bmatrix} a_4 \\ x_4 \end{bmatrix} \begin{bmatrix} a_{11} \\ x_{11} \end{bmatrix} \begin{bmatrix} a_{12} \\ x_{12} \end{bmatrix} \begin{bmatrix} a_{13} \\ x_{13} \end{bmatrix}$$

Furthermore, we have $x' + x'' = 4 = 1$. Hence either $x' = x'' = 2$ or $x' = 0$ or $x'' = 0$. If $x' = 0$ or $x'' = 0$ we are through by 11.

It is easy to see that any element $\begin{bmatrix} a_i \\ 1 \end{bmatrix}$ $(i=0,\dots,15)$ is the sum of a subsequence of length 1, 4 or 7 of the 7 remaining elements. This guarantees, for every value of a, the existence of a zero-sequence in S.

18. <u>CASUS PERDIFFICILIS, $n \geq 5$.</u>
This case is treated as the case (2) of 17. If there are ≥ 2 short zero-sequences in A of length 1 we see that A contains $\geq n$ disjoint zero-subsequences and we are done.

As A contains $n - 2 > 2$ short zero-sequences we certainly have a zero-sequence of length 2 in A; this short sequence we join to the remainder of length 7; thus we write A as a collection of $n - 3$ short zero-sequences, and a collection of 9 remaining elements, of the following type:

$$(*) \qquad \begin{bmatrix} a_1 \\ 1 \end{bmatrix}, \begin{bmatrix} a_2 \\ 1 \end{bmatrix}, \begin{bmatrix} a_3 \\ 1 \end{bmatrix}, \begin{bmatrix} a_4 \\ 1 \end{bmatrix}, \begin{bmatrix} a_{11} \\ 1 \end{bmatrix}, \begin{bmatrix} a_{12} \\ 1 \end{bmatrix}, \begin{bmatrix} a_{13} \\ 1 \end{bmatrix}, \begin{bmatrix} a \\ x' \end{bmatrix}, \begin{bmatrix} a \\ x'' \end{bmatrix}.$$

We have $x' + x'' = 4$. Furthe the x-sum of any subsequence with A-sum a_0 is equal 4 or 8, the latter value being excluded if the subsequence belongs to a pair of disjoint subsequences with A-sum a_0.

Now the element $-a$ is the sum of ≤ 3 elements from the collection $\{a_1, a_2, a_3, a_4, a_{11}, a_{12}, a_{13}\}$. Say $a_{i_1}, \ldots, a_{i_r}$.

Therefore $\begin{bmatrix} a_{i_1} \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} a_{i_r} \\ 1 \end{bmatrix}, \begin{bmatrix} a \\ x' \end{bmatrix}$ is a sequence of length ≤ 4 with

A-sum a_0. The remaining ≥ 5 elements contain another sequence with A-sum a_0 hence $x' + 1 + \ldots + 1 = 4$. Similarly we have $x'' + 1 + \ldots + 1 = 4$. Therefore $x' = x''$, which implies $x' = 2 = x''$.

If $a \in \{a_1, a_2, a_3, a_4, a_{11}, a_{12}, a_{13}\}$, then (*) contains a subsequence with sum $\begin{bmatrix} a_0 \\ 3 \end{bmatrix}$ and length 2. As $3 \not\equiv 4 \pmod n$ we are done.

If $a = a_{14}$ then we have

$$\begin{bmatrix} a_2 \\ 1 \end{bmatrix} + \begin{bmatrix} a_3 \\ 1 \end{bmatrix} + \begin{bmatrix} a_4 \\ 1 \end{bmatrix} + \begin{bmatrix} a \\ 2 \end{bmatrix} = \begin{bmatrix} a_0 \\ 5 \end{bmatrix}$$

We have also $a_1 + a_2 + a_3 + a_{11} = a_0$. As $5 \not\equiv 4 \bmod n$ we are through in this case also.

If $a = a_{15}$ we have

$$\begin{bmatrix} a_1 \\ 1 \end{bmatrix} + \begin{bmatrix} a_2 \\ 1 \end{bmatrix} + \begin{bmatrix} a_3 \\ 1 \end{bmatrix} + \begin{bmatrix} a_4 \\ 1 \end{bmatrix} + \begin{bmatrix} a_{15} \\ 2 \end{bmatrix} = \begin{bmatrix} a_0 \\ 6 \end{bmatrix}$$

This is impossible as $4 \not\equiv 6 \not\equiv 8 \pmod n$.

The remaining cases are $a = a_5, a_6, a_7, a_8, a_9, a_{10}$. By symmetry the first three and the latter three of these are equivalent.
$a = a_5$ implies that S contains a zero-subsequence as

$$\begin{bmatrix} a_2 \\ 1 \end{bmatrix} + \begin{bmatrix} a_3 \\ 1 \end{bmatrix} + \begin{bmatrix} a_4 \\ 1 \end{bmatrix} + \begin{bmatrix} a_{13} \\ 1 \end{bmatrix} + \begin{bmatrix} a \\ 2 \end{bmatrix} = \begin{bmatrix} a_0 \\ 6 \end{bmatrix} \notin \left\{ \begin{bmatrix} a_0 \\ 4 \end{bmatrix}, \begin{bmatrix} a_0 \\ 8 \end{bmatrix} \right\},$$

and $a = a_8$ leads to

$$\begin{bmatrix} a_1 \\ 1 \end{bmatrix} + \begin{bmatrix} a_2 \\ 1 \end{bmatrix} + \begin{bmatrix} a_4 \\ 1 \end{bmatrix} + \begin{bmatrix} a_{13} \\ 1 \end{bmatrix} + \begin{bmatrix} a \\ 2 \end{bmatrix} = \begin{bmatrix} a_0 \\ 6 \end{bmatrix} \notin \left\{ \begin{bmatrix} a_0 \\ 4 \end{bmatrix}, \begin{bmatrix} a_0 \\ 8 \end{bmatrix} \right\} ,$$

which again implies the existence of such a zero-sequence. This completes the treatment of the CASUS PERDIFFICILIS.

19. Final remarks.

If the dimension of the group is enlarged by 1 a situation similar to the CASUS PERDIFFICILIS leads to an example of a group G for which the statement G! is false. The smallest example of such a group (with respect to the number of elements) is the group $G = C_2 \oplus C_2 \oplus C_2 \oplus C_2 \oplus C_6$.

Just consider

$$\begin{array}{c|cccccccccc}
C_2 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
C_2 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
C_2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
C_2 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
C_2 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
C_3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array} \quad ;$$

this is a sequence of length $10 > \lambda(G) = 9$ which contains no zero-subsequence.

The calculation above tempts one to expect that the statement $(C_2 \oplus C_2 \oplus C_2 \oplus C_2 \oplus C_{2n})!$ will be true for $n \geq 5$. A verification of this conjecture, however, will probably be time-consuming.

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